

EQUIVARIANT EULER CHARACTERISTICS OF PARTITION POSETS

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ABSTRACT. We compute all the equivariant Euler characteristics of the Σ_n -poset of partitions of the n element set.

1. INTRODUCTION

Let G be a finite group and Π a finite G -poset. For $r \geq 1$, let $C_r(G)$ denote the set of tuples $X = (x_1, \dots, x_r)$ of r commuting elements of G . Write Π^X for the subposet consisting of all elements of Π fixed by all elements of $X \in C_r(G)$. The r th reduced equivariant Euler characteristic of the G poset Π , as defined by Atiyah and Segal [2], is the normalized sum

$$\tilde{\chi}_r(\Pi, G) = \frac{1}{|G|} \sum_{X \in C_r(G)} \tilde{\chi}(\Pi^X)$$

of the reduced Euler characteristics of the subposets Π^X as X runs through the set $C_r(G)$ of commuting r -tuples.

In this note we focus on equivariant Euler characteristics of partition posets. Let Σ_n denote the symmetric group of degree $n \geq 2$. The set $\Pi(\Sigma_{n-1} \setminus \Sigma_n)$ of partitions of the standard right Σ_n -set $\Sigma_{n-1} \setminus \Sigma_n$ is a (contractible) right Σ_n -lattice with smallest element $\hat{0}$, the discrete partition, and largest element $\hat{1}$, the indiscrete partition. We let $\Pi^*(\Sigma_{n-1} \setminus \Sigma_n) = \Pi(\Sigma_{n-1} \setminus \Sigma_n) - \{\hat{0}, \hat{1}\}$ be the (non-contractible) Σ_n -poset obtained by removing $\hat{0}$ and $\hat{1}$.

We now state the result and defer the the explanation of the undefined expressions till after theorem.

Theorem 1.1. *The r th reduced equivariant Euler characteristic of the Σ_n -poset $\Pi^*(\Sigma_{n-1} \setminus \Sigma_n)$ of partitions is*

$$\tilde{\chi}_r(\Pi^*(\Sigma_{n-1} \setminus \Sigma_n), \Sigma_n) = \frac{1}{n} (a * b_r)(n)$$

when $n \geq 2$ and $r \geq 1$.

The multiplicative arithmetic sequence $a * b_r$ is the Dirichlet convolution

$$(a * b_r)(n) = \sum_{d_1 d_2 = n} a(d_1) b_r(d_2)$$

of the the multiplicative arithmetic sequences a and b_r given by

$$a(n) = (-1)^{n+1}, \quad b_r(n) = \prod_p (-1)^{n_p} p^{\binom{n_p}{2}} \binom{r}{n_p}_p, \quad r \geq 1, n \geq 1$$

where $n = \prod_p p^{n_p}$ is the prime factorization of n and the p -binomial coefficient

$$\binom{r}{d}_p = \frac{(p^r - 1) \cdots (p^r - p^{d-1})}{(p^d - 1) \cdots (p^d - p^{d-1})}$$

is the number of d -dimensional subspaces of the r -dimensional \mathbf{F}_p -vector space [9, Proposition 1.3.18].

2. PARTITIONS OF FINITE G -SETS

Let G be a group and S a finite right G -set.

- Definition 2.1.** (1) A partition π of S is an equivalence relation on S . The blocks of π are the equivalence classes of π . For any $x \in S$, $[x]_\pi$, or simply $[x]$, is the π -block of x . The set of π -blocks is denoted $\pi \setminus S$.
(2) $\Pi(S)$ is the G -lattice of all partitions of S and $\Pi^*(S) = \Pi(S) - \{\hat{0}, \hat{1}\}$ the G -poset of all partitions of S but the discrete and the indiscrete partitions, $\hat{0}$ and $\hat{1}$.
(3) A partition of S is a G -partition if $x \sim y \iff xg \sim yg$ for all $x, y \in S$ and $g \in G$.

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- (4) $\Pi(S)^G$ is the lattice of all G -partitions of S and $\Pi^*(S)^G = \Pi(G)^G - \{\widehat{0}, \widehat{1}\}$ the poset of all G -partitions of S but the discrete and indiscrete partitions.
- (5) The isotropy subgroup at $x \in S$ is the subgroup ${}_xG = \{g \in G \mid xg = x\}$ of G .
- (6) If π is a G -partition, the block isotropy subgroup at $x \in S$ is the isotropy subgroup ${}_{[x]}G = \{g \in G \mid (xg)\pi x\}$ at the π -block $[x]$ of x in the G -set $\pi \backslash S$ of π -blocks.
- (7) The G -set S is isotypical if all isotropy subgroups are conjugate.
- (8) The G -partition $\pi \in \Pi(S)^G$ is isotypical if the G -set $\pi \backslash S$ of π -blocks is isotypical. $\Pi^{\text{iso}}(S)^G$ is the poset of all isotypical G -partitions and $\Pi^{*+\text{iso}}(S)^G = \Pi^{\text{iso}}(S)^G - \{\widehat{0}, \widehat{1}\}$.

The set $\Pi(S)$ of partitions of S is partially ordered by refinement:

$$\pi_1 \leq \pi_2 \iff \forall x \in S: [x]_{\pi_1} \subseteq [x]_{\pi_2}$$

The meet of π_1 and π_2 is the partition $\pi_1 \wedge \pi_2$ with blocks $[x]_{\pi_1 \wedge \pi_2} = [x]_{\pi_1} \cap [x]_{\pi_2}$, $x \in S$. The discrete partition is $\widehat{0}$ with blocks $[x]_{\widehat{0}} = \{x\}$, $x \in S$, and the indiscrete partition is $\widehat{1}$ with block $[x]_{\widehat{1}} = S$, $x \in S$.

Example 2.2. Let K be a subgroup of G . The partition ω_K , whose blocks $[x]_{\omega_K} = xK$ are the K -orbits in S , is an $N_G(K)$ -partition of S . In particular, the partition ω_G whose blocks are the G -orbits is a G -partition.

The set $\Pi(S)$ of partitions of S is a right G -lattice: For any partition π of S and any $g \in G$, πg is the partition given by $x(\pi g)y \iff (xg)\pi(yg)$. Then $[x]_{\pi g} = \{yg \mid x(\pi g)y\} = \{yg \mid (yg)\pi(xg)\} = \{y \mid y\pi(xg)\} = [xg]_{\pi}$. Obviously,

$$\pi \text{ is a } G\text{-partition} \iff \forall g \in G: \pi g = \pi \iff \forall g \in G \forall x \in X: [x]_{\pi g} = [xg]_{\pi} \iff \forall g \in G \forall b \in \pi: bg \in \pi$$

Thus the fixed poset for this G -action on $\Pi(S)$, $\Pi(S)^G$, is the set of all G -partitions. The discrete and the indiscrete partitions are G -partitions.

Proposition 2.3. Let π be a G -partition of S .

- (1) There is a right G -action on the set $\pi \backslash S$ of π -blocks such that $S \rightarrow \pi \backslash S$ is a G -map.
- (2) ${}_xG \leq {}_{[x]}G$ for any $x \in S$.
- (3) ${}_{xg}G = {}_xG^g$ and ${}_{[xg]}G = {}_{[x]}G^g$
- (4) ${}_{xg}G \leq {}_{[x]}G^g$ for any $x \in S$ and any $g \in G$.

Proof. The G -action on $\pi \backslash S$ is given by $[x]g = [xg]$ for all $x \in S$ and $g \in G$. □

Definition 2.4. Let P be a subposet of a lattice. An element c of P is a *contractor* if $x \vee c \in P$ or $x \wedge c \in P$ for all $x \in P$.

If c is a contractor for P then $x \leq x \vee c \geq c$ or $x \geq x \wedge c \leq c$ are homotopies between the identity map of P and the constant map c . We view P as a finite topological space with the order right ideals as open sets.

Lemma 2.5. [1, Lemma 7.1] $\Pi^*(S)^G$ is contractible unless S is isotypical.

Proof. Let ω_G be the G -partition represented by the G -map $S \rightarrow S/G$ to the G -set of G -orbits and θ_G the G -partition represented by the G -map $S \rightarrow S/G \rightarrow \cong \backslash S/G$ to the set of isomorphism classes of G -orbits. Explicitly, $x\omega_G y$ if and only if x and y are in the same G -orbit, and $x\theta_G y$ if and only if x and y have conjugate isotropy subgroups. We shall prove that θ_G is a contractor (Definition 2.4) for $\Pi^*(S)^G$ when S is not isotypical.

We first make some small observations. Obviously, $\omega_G \leq \theta_G$. The G -action is trivial if and only if $\omega_G = \widehat{0}$. The G -action is isotypical if and only if $\theta_G = \widehat{1}$. If the G -action is trivial, all isotropy subgroups are equal to G , and therefore $\theta_G = \widehat{1}$. We may summarize these observations in a string

$$\theta_G = \widehat{0} \implies \omega_G = \widehat{0} \iff \forall x \in S: {}_xG = G \implies \theta_G = \widehat{1} \iff S \text{ is isotypical}$$

of implications.

Let π be any G -partition of S . We claim that

$$(2.6) \quad \pi \wedge \theta_G = \widehat{0} \implies \pi = \widehat{0}$$

To see this first note that

$$\forall x, y \in S: x\pi y \implies y \cdot {}_xG \subseteq [y]_{\pi \wedge \theta_G}$$

Indeed, let $x\pi y$ and $g \in {}_xG$. Then $y\pi(yg)$ for $y\pi x$, $x = xg$, and $(xg)\pi(yg)$. Thus y and yg are both in $[y]_{\pi}$ and in $[y]_{\theta_G}$. Now assume that $\pi \wedge \theta_G = \widehat{0}$. Then

$$\forall x, y \in S: x\pi y \implies {}_xG \leq {}_yG$$

for the block $[y]_{\pi \wedge \theta_G} = [y]_{\widehat{0}} = \{y\}$ consists of y alone which forces $yg = y$ for all $g \in {}_x G$. This can be sharpened to

$$\forall x, y \in S: x\pi y \implies {}_x G = {}_y G$$

as the equivalence relation π is symmetric, of course. Now, when x and y have the same isotropy subgroups, x and y belong to the same block under θ_G . Thus we have shown $\pi \leq \theta_G$. Then $\pi = \pi \wedge \theta_G = \widehat{0}$. This proves claim (2.6).

Suppose that S is not isotypical. Then $\theta_G \neq \widehat{0}, \widehat{1}$ and θ_G belongs to the poset $\Pi^*(S)^G$. From claim (2.6) we know that $\pi \wedge \theta_G \neq \widehat{0}$ for all $\pi \in \Pi^*(S)^G$. Thus θ_G is a contractor for $\Pi^*(S)^G$. \square

There are, of course, isotypical G -sets S for which $\Pi^*(S)^G$ is contractible.

Example 2.7 (An isotypical G -set S such that $\Pi^*(S)^G$ is contractible). Suppose that the Frattini subgroup $\Phi(G)$ of G is nontrivial and proper. The G -set $S = G$ is transitive and hence isotypical. But still the poset $\Pi^*(S)^G$ is contractible: By Proposition 2.8, $\Pi^*(S)^G$ is the poset $(1, G)$ of non-identity proper subgroups of G , and $\Phi(G)$ is a contractor of $(1, G)$. (I thank Matthew Gelvin for pointing out this example.)

A G -partition of a transitive G -set S is uniquely determined by its block isotropy subgroup at a single point.

Proposition 2.8. [10, Lemma 3] *Let S be a transitive G -set and x a point of S . The block isotropy map*

$$\Pi(S)^G \rightarrow [{}_x G, G] = {}_x G / S_G: \pi \rightarrow [x]_{\pi} G$$

is an isomorphism of posets.

Proof. Let $H = {}_x G$ be the isotropy subgroup of x . For every subgroup K of G containing H , let π_K be the G -partition of S with blocks xKg , $g \in G$ (the fibres of $S = H \backslash G \rightarrow K \backslash G$). The π_K -block of x , $[x]_{\pi_K} = xK$, has isotropy subgroup $\{g \in G \mid xg \in xK\} = K$. Conversely, let π be any G -partition of S . The orbit through x of the block isotropy subgroup $[x]_{\pi} G$ is $x \cdot [x]_{\pi} G = [x]_{\pi}$ as S is transitive. These observations show that $K \rightarrow \pi_K$ is an inverse to the block isotropy subgroup map $\pi \rightarrow [x]_{\pi} G$. It is clear that these bijections respect the partial orderings. \square

Definition 2.9. \mathcal{O}_G is the category of finite G -sets with surjective G -maps as morphisms.

We may consider G -partitions as morphisms in the category \mathcal{O}_G . To any G -partition π of the G -set S we associate the surjective G -map $S \rightarrow \pi \backslash S$. Conversely, the blocks of the partition represented by the surjective G -map $\pi: S \rightarrow T$ are the fibres of π . The block of $x \in S$ is $\pi^{-1}(\pi(x))$. The overlap of the block and the G -orbit of x is the orbit through x of the block isotropy subgroup, $\pi^{-1}(\pi(x)) \cap xG = x_{\pi(x)} G$.

3. EULER CHARACTERISTICS OF POSETS OF G -PARTITIONS

Let Π be a finite poset. For $a, b \in \Pi$ let

$$\begin{aligned} a/\Pi &= \{p \in \Pi \mid a \leq p\} & a//\Pi &= \{p \in \Pi \mid a < p\} & k^a &= -\widetilde{\chi}(a//\Pi) \\ \Pi/b &= \{p \in \Pi \mid p \leq b\} & \Pi//b &= \{p \in \Pi \mid p < b\} & k_b &= -\widetilde{\chi}(\Pi//b) \end{aligned}$$

denote the coslice of Π under a , the proper coslice of Π under a , and the weighting at a , and, dually, the slice of Π over b , the proper slice of Π over b , and the coweighting at b [4, Corollary 3.8]. The Euler characteristic of Π

$$\sum_{a \in \Pi} k^a = \chi(\Pi) = \sum_{b \in \Pi} k_b$$

is the sum of the values of the weighting or coweighting. In particular, for a finite G -set S , we can compute the Euler characteristic of $\Pi^*(S)^G$,

$$(3.1) \quad \sum_{\pi \in \Pi^*(S)^G} -\widetilde{\chi}(\pi//\Pi^*(S)^G) = \chi(\Pi^*(S)^G) = \sum_{\pi \in \Pi^*(S)^G} -\widetilde{\chi}(\Pi^*(S)^G//\pi)$$

from its weighting or coweighting [4, Corollary 3.8]. We shall now determine these functions.

Proposition 3.2 (Slices in $\Pi^*(S)^G$). *For any G -partition π of the right G -set S*

$$\pi/\Pi(S)^G = \Pi(\pi \backslash S)^G, \quad \pi//\Pi^*(S)^G = \Pi^*(\pi \backslash S)^G$$

The weighting for $\Pi^(S)^G$*

$$k^{\pi} = -\widetilde{\chi}(\Pi^*(\pi \backslash S)^G), \quad \pi \in \Pi^*(S)^G,$$

vanishes at π unless π is isotypical (Definition 2.1.(8)).

Proof. Let ρ be a partition of the right G -set $\pi \backslash G$ of blocks of π . There is then a partition of S with blocks $[x] = [[x]_{\pi}]_{\rho}$, $x \in S$. This new partition is a G -partition if and only if ρ is a G -partition of $\pi \backslash S$. Any G -partition $\geq \pi$ of S arises in this way. \square

Proposition 3.3 (Coslices in $\Pi^*(S)^G$). *For any G -partition π of the right G -set S*

$$\Pi(S)^G/\pi = \prod_{BG \in \pi \backslash S/G} \Pi(B)^{BG}, \quad \Pi^*(S)^G//\pi = \left(\prod_{BG \in \pi \backslash S/G} \Pi(B)^{BG} \right)^*$$

The coweighting for $\Pi^(S)^G$*

$$k_\pi = - \prod_{\substack{BG \in \pi \backslash S/G \\ |B| > 1}} \tilde{\chi}(\Pi^*(B)^{BG}), \quad \pi \in \Pi^*(S)^G,$$

vanishes at π unless all blocks B of π are isotypical BG -sets.

Proof. Let π be a G -partition and B one its blocks. Observe first that the blocks contained in B of a G -partition $\lambda \leq \pi$ determine all blocks of λ contained in any of the blocks of the orbit BG through B for the G -action on $\pi \backslash S$.

Let B be a block, with isotropy subgroup ${}_BG$, of the G -partition π . Let λ be a ${}_BG$ partition of B . Extend λ to a G -partition of the orbit BG of B in π by $[xg]_\lambda = [x]_\lambda g$. We must argue that this extension is well-defined. Suppose that $x_1 g_1 = x_2 g_2$ for some $x_1, x_2 \in B$ and $g_1, g_2 \in G$. We must show that $[x_1]_\lambda g_1 = [x_2]_\lambda g_2$. We have $x_2 = x_2 g_2 g_2^{-1} = x_1 g_1 g_2^{-1}$. From $B = [x_2]_\pi = [x_1 g_1 g_2^{-1}]_\pi = [x_1]_\pi g_1 g_2^{-1} = B g_1 g_2^{-1}$ we get that $g_1 g_2^{-1}$ stabilizes the block B . As λ is a ${}_BG$ -partition, $[x_1]_\lambda g_1 = [x_1]_\lambda g_1 g_2^{-1} g_2 = [x_1 g_1 g_2^{-1}]_\lambda g_2 = [x_2]_\lambda g_2$ as we wanted.

Conversely, if λ is a G -partition and $\lambda \leq \pi$ then the blocks of λ inside a fixed block B of π form a ${}_BG$ -partition of B , of course.

According to Quillen the reduced Euler characteristic is multiplicative: $\tilde{\chi}((\prod L_i)^*) = \prod \tilde{\chi}(L_i^*)$ for lattices L_i of more than one element [1, Proposition 2.8].

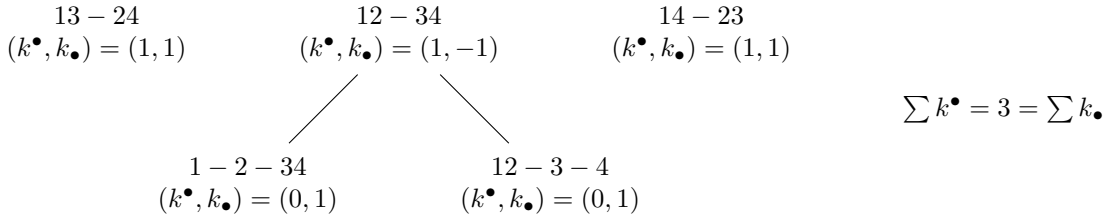
If the block B of partition π consists of a single element of S , then also the partition poset $\Pi(B)$ consists of a single element so it can be omitted from the poset product $\prod_{B \in \pi \backslash S} \Pi(B)$. \square

In all cases,

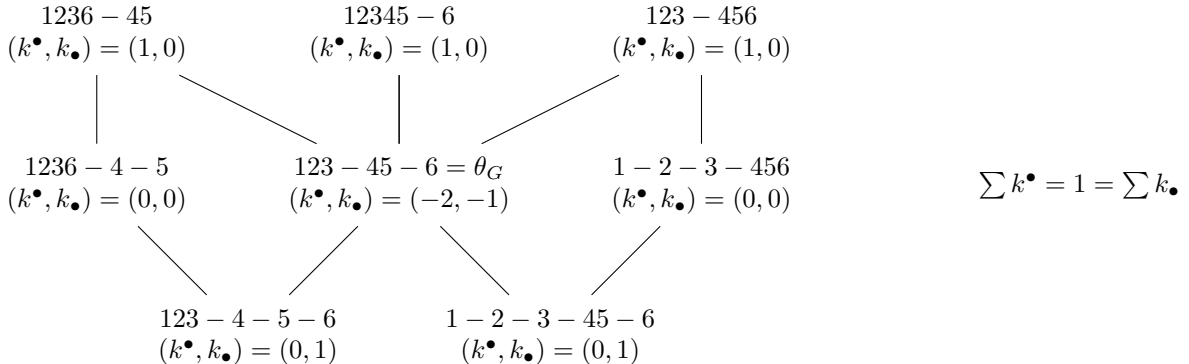
$$(3.4) \quad \sum_{\pi \in \Pi^*(S)^G} \tilde{\chi}(\Pi^*(\pi \backslash S)^G) = -\chi(\Pi^*(S)^G) = \sum_{\pi \in \Pi^*(S)^G} \prod_{\substack{BG \in \pi \backslash S/G \\ |B| > 1}} \tilde{\chi}(\Pi^*(B)^{BG})$$

where the sum on the left can be restricted to the G -partitions π with G -isotypical block set $\pi \backslash S$, and the sum on the right can be restricted to the G -partitions π for which ${}_BG$ acts isotypically on every block B of π . If G acts non-isotypically on S then these sums equal 0.

Example 3.5 (Two examples of G -partition posets). The poset $\Pi^*(S)^G$ of nontrivial G -partitions for $S = \{1, 2, \dots, 4\}$ and $G = \langle (1, 2)(4, 5) \rangle \leq \Sigma_4$ (isotypical):



The poset $\Pi^*(S)^G$ of nontrivial G -partitions for $S = \{1, 2, \dots, 6\}$ and $G = \langle (1, 2, 3), (4, 5) \rangle \leq \Sigma_6$ (non-isotypical):



Corollary 3.6. *The inclusion $\Pi^{*+iso}(S)^G \hookrightarrow \Pi^*(S)^G$ is a homotopy equivalence.*

Proof. This follows immediately from Bouc's theorem [3] since $\pi//\Pi^*(S)^G$ is contractible unless π is isotypical by Proposition 3.2 and Lemma 2.5. \square

Because of Corollary 3.6 we now restrict attention to isotypical G -partitions of isotypical G -sets.

For any G -orbit S and any natural number $n \geq 1$, let $nS = \coprod_n S$ be the isotypical G -set with n G -orbits isomorphic to S .

Definition 3.7. Let S and T be G -orbits.

- An nS/kT -partition is an isotypical G -partition of nS with block G -set isomorphic to kT .
- The G -Stirling number of the second kind

$$S_G(nS, kT) = |\{\pi \in \Pi(nS)^G \mid \pi \setminus (nS) \cong kT\}|$$

is the number nS/kT -partitions.

In the following, \mathcal{S}_G is the poset of subgroups, and $[\mathcal{S}_G]$ the poset of subgroup conjugacy classes of G . We write ζ_G , or just ζ , for the poset incidence matrix (with $\zeta_G(H, K) = 1$ if $H \leq K$ and $\zeta_G(H, K) = 0$ otherwise) and $\mu = \mu_G = \zeta_G^{-1}$ for the Möbius matrix of \mathcal{S}_G .

Definition 3.8. The G -Stirling matrix of degree n is the square $(n|[\mathcal{S}_G]| \times n|[\mathcal{S}_G]|)$ -matrix

$$[\zeta]_G \otimes S_G = ((S_G(sH \setminus G, tK \setminus G))_{1 \leq s, t \leq n})_{H, K \in [\mathcal{S}_G]}$$

obtained as the $(|[\mathcal{S}_G]| \times |[\mathcal{S}_G]|)$ -matrix of $(n \times n)$ -block matrices $(S_G(sH \setminus G, tK \setminus G))_{1 \leq s, t \leq n}$ of Stirling numbers with fixed G -orbits $G \setminus H$ and $K \setminus G$.

If we order the subgroups of G in decreasing order starting with G itself, the G -Stirling matrix is lower triangular.

If we in Equation 3.1 insert the values from Proposition 3.2 we obtain formulas for the reduced Euler characteristic of the poset $\Pi^*(nS)^G$,

$$(3.9) \quad \tilde{\chi}(\Pi^*(nS)^G) = -1 - \sum_{T, k} \tilde{\chi}(\Pi^*(kT)^G) S_G(nS, kT), \quad 1 = \sum_{k|T| > 1} -\tilde{\chi}(\Pi^*(kT)^G) S_G(nS, kT)$$

with T ranging over the set of isomorphism classes of G -orbits and $k \geq 1$ over natural numbers with $k|T| > 1$. (Observe that $S_G(nS, nS) = 1$.) In matrix notation

$$(3.10) \quad (S_G(sH \setminus G, tK \setminus G))_{\substack{H, K \in [\mathcal{S}_G] \\ 1 \leq s, t \leq n}} \begin{pmatrix} \vdots \\ -\tilde{\chi}(\Pi^*(sH \setminus G)^G) \\ \vdots \end{pmatrix}_{\substack{S \in [\mathcal{S}_G] \\ 1 \leq s \leq n}} = \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 1 \end{pmatrix}$$

we see that minus the reduced Euler characteristics of the G -partitions of the isotypical G -sets are a weighting for the Stirling matrix of G . Equation (3.10) comes with the caveat that the top entry of the left column vector is 0 and not $-\tilde{\chi}(\Pi^*(1G \setminus G)^G) = 1$.

Example 3.11 (G -Stirling matrices of degree 1). The Stirling number for single orbits $S = H \setminus G$ and $T = K \setminus G$,

$$S_G(H \setminus G, K \setminus G) = S_G(H, [K]) = \frac{|N_G(H, K)|}{|N_G(K, K)|} = \frac{|\mathcal{S}_G(H \setminus G, K \setminus G)|}{|\mathcal{S}_G(K \setminus G)|} = \frac{|(K \setminus G)^H|}{|(K \setminus G)^K|} = \frac{\text{TOM}(H, K)}{\text{TOM}(K, K)}$$

is the number, $\mathcal{S}_G(H, [K]) = |\{L \in [K] \mid H \leq L\}|$, of conjugates of K containing H [7, Definition 3.5, Lemma 3.6]. This number is determined by the table of marks $\text{TOM}(H, K) = |(K \setminus G)^H|$ for G . Proposition 2.8 or [7] show that the entries of the column vector in Equation (3.10) are

$$-\tilde{\chi}(\Pi^*(H \setminus G)^G) = -\tilde{\chi}(H, G) = -\mu(H, G)$$

for all proper subgroups H of G . (In any finite poset, $\mu(x, y) = \tilde{\chi}(x, y)$ whenever $x < y$ [9, Proposition 3.8.5].)

For instance, $G = \Sigma_3$ has $|[\mathcal{S}_{\Sigma_3}]| = 4$ orbits S_1, S_2, S_3, S_6 of sizes 1, 2, 3, 6. The Σ_3 -Stirling matrix of degree 1 is

$S_{\Sigma_3}(S, T)$	$\Sigma_3 \setminus \Sigma_3$	$A_3 \setminus \Sigma_3$	$C_2 \setminus \Sigma_3$	$C_1 \setminus \Sigma_3$	$-\tilde{\chi}(\Pi^*(H \setminus \Sigma_3)^{\Sigma_3})$
$\Sigma_3 \setminus \Sigma_3$	1				0
$A_3 \setminus \Sigma_3$	1	1			1
$C_2 \setminus \Sigma_3$	1	0	1		1
$C_1 \setminus \Sigma_3$	1	1	3	1	-3

and (remembering the caveat that the top entry of the column to the far right is 0 when solving Equation (3.10)) we read off that $\mu(A_3, \Sigma_3) = -1$, $\mu(C_2, \Sigma_3) = -1$, $\mu(1, \Sigma_3) = 3$.

Since $\tilde{\chi}(\Pi^*(1H \setminus G)^G) = \tilde{\chi}(H, G) = \mu(H, G)$ for proper subgroups H of G by Proposition 2.8, it seems natural to define the higher Möbius numbers to be the solutions to the linear equation (3.10).

Definition 3.12 (Higher Möbius numbers). *For every subgroup H of G and every natural number $n \geq 1$ let*

$$\mu_n(H, G) = \tilde{\chi}(\Pi^*(nH \setminus G)^G)$$

with the convention that $\mu_1(G, G) = 1$.

For any group G , $\mu_n(G, G) = (-1)^n(n-1)! = \mu_n(1, 1)$ for $n \geq 2$, and $\mu_n(1, G) = \tilde{\chi}(\Pi^*(\coprod_n G)^G)$ for $n \geq 1$. With $n = 1$, $\mu_1(H, G) = \mu(H, G)$ is the usual Möbius function of \mathcal{S}_G as considered in Example 3.11.

The higher Möbius numbers $\mu_n(H, G)$ for $1 \leq h \leq n$ are determined by the G -Stirling matrix of degree n . We shall now consider the problem of determining the entries of this matrix.

Let $S(n, k)$ stand both for the poset of partitions of the n element set with k blocks and for the Stirling number (Example 3.17) of such partitions. Then

$$S_G(nH \setminus G, kK \setminus G) = \sum_{\pi \in S(n, k)} \prod_{b \in \pi} \frac{|\mathcal{O}_G(H \setminus G, K \setminus G)|^{|b|}}{|\mathcal{O}_G(K \setminus G, K \setminus G)|} = \frac{|\mathcal{O}_G(H \setminus G, K \setminus G)|^n}{|\mathcal{O}_G(K \setminus G, K \setminus G)|^k} S(n, k) = \frac{\text{TOM}(H, K)^n}{\text{TOM}(K, K)^k} S(n, k)$$

In particular

$$(3.13) \quad S_G(nS, kT) = \begin{cases} |T|^{n-k} S(n, k) & \mathcal{O}_G(S, T) \neq \emptyset \\ 0 & \mathcal{O}_G(S, T) = \emptyset \end{cases}$$

when G is abelian.

Lemma 3.14. *If $H \trianglelefteq G$ is normal in G , then $\mu_n(H, G) = \mu_n(1, H \setminus G)$ for all $n \geq 1$.*

Proof. H acts trivially on $H \setminus G$ as $Hgh = Hghg^{-1}g = Hg$ for all $h \in H, g \in G$. Thus a partition of $nH \setminus G$ is a G -partition if and only if it is a $H \setminus G$ -partition. \square

The higher Möbius numbers $\mu_1(H, G), \dots, \mu_n(H, G)$ for $H \leq G$ (except for $\mu_1(G, G)$ which by decree equals 1) solve the system of linear equations (3.10) which we now rewrite as

$$(3.15) \quad [\zeta]_G \otimes S_G \begin{bmatrix} 0 \\ -\mu_2(G, G) \\ \vdots \\ -\mu_n(G, G) \\ \vdots \\ -\mu_1(H, G) \\ \vdots \\ -\mu_n(H, G) \\ \vdots \\ -\mu_1(1, G) \\ \vdots \\ -\mu_n(1, G) \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 1 \\ \vdots \\ 1 \\ \vdots \\ 1 \\ \vdots \\ 1 \\ \vdots \\ 1 \end{bmatrix}$$

with the G -Stirling matrix as coefficient matrix. We shall adapt the convention that in the Stirling matrix the groups will be listed with decreasing order. The group G itself occurs as the first group in the Stirling matrix which is lower triangular. The first n columns are made up of the block matrices $(S(i, j))_{1 \leq i, j \leq n}$ of classical Stirling numbers. All entries of the first column, in particular, equal $S(n, 1) = 1$. Thus

$$[\zeta]_G \otimes S_G \begin{bmatrix} \mu_1(G, G) \\ \vdots \\ \mu_n(G, G) \\ \vdots \\ \mu_1(1, G) \\ \vdots \\ \mu_n(1, G) \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

or

$$(3.16) \quad \begin{bmatrix} \mu_1(G, G) \\ \vdots \\ \mu_n(G, G) \\ \vdots \\ \mu_1(1, G) \\ \vdots \\ \mu_n(1, G) \end{bmatrix} = ([\zeta]_G \otimes S_G)^{-1} \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

The entries of the inverted matrix $([\zeta]_G \otimes S_G)^{-1}$ are the G -Stirling numbers of the first kind [9, p 36].

Example 3.17 (Higher Möbius numbers of the trivial group). The C_1 -Stirling matrix of the second kind (Definition 3.7) is the matrix

$$S = \begin{bmatrix} 1 & & & & & \\ 1 & 1 & & & & \\ 1 & 3 & 1 & & & \\ 1 & 7 & 6 & 1 & & \\ 1 & 15 & 25 & 10 & 1 & \\ 1 & 31 & 90 & 65 & 15 & 1 \end{bmatrix}$$

of classical Stirling numbers $S(n, k) = |\{\pi \in \Pi_n \mid |\pi| = k\}|$ of the second kind. The higher Möbius numbers of the trivial group are by Equation (3.16) equal to the Stirling numbers of the first kind [9, p 36]

$$\mu_n(1, 1) = (S^{-1})(n, 1) = s(n, 1) = (-1)^{n-1}(n-1)!, \quad n \geq 1$$

We have re-derived the classical formula [9, Example 3.10.4] for the reduced Euler characteristic of the partition poset.

Lemma 3.18. *If the group G is abelian then*

$$\mu_n(H, G) = \mu(H, G)|G : H|^{n-1}\mu_n(1, 1)$$

for all $n \geq 1$ and all subgroups $H \leq G$.

Proof. Since G is abelian, $S_G(iH \setminus G, jK \setminus G) = |G : K|^{i-j}S(i, j)$ by Equation (3.13), and the G -Stirling matrix of degree n is the block matrix

$$((\zeta(H, K)|G : K|^{i-j}S(i, j))_{1 \leq i, j \leq n})_{H, K \in [S_G]}$$

The vector $((\mu_i(H, G))_{1 \leq i \leq n})_{H \in S_G}$ is (Equation (3.16)) the first column

$$([\mu](H, K)|G : H|^{i-1}S^{-1}(i, 1))_{1 \leq i \leq n})_{H \in [S_G]} = (\mu(H, K)|G : H|^{i-1}\mu_i(1, 1))_{1 \leq i \leq n})_{H \in [S_G]}$$

in the inverse matrix

$$((\mu(H, K)|G : H|^{i-j}S^{-1}(i, j))_{1 \leq i, j \leq n})_{H, K \in [S_G]}$$

of the G -Stirling matrix. □

In the example below we consider an example of a Stirling matrix for a non-abelian group.

Example 3.19. The Σ_3 -Stirling matrix of degree 3 (reusing the notation of Example 3.11) is

$S_{\Sigma_3}(S, T)$	$1S_1$	$2S_1$	$3S_1$	$1S_2$	$2S_2$	$3S_2$	$1S_3$	$2S_3$	$3S_3$	$1S_6$	$2S_6$	$3S_6$	$-\mu_i(H, \Sigma_3)$
$1S_1$	1	0	0										0
$2S_1$	1	1	0										1
$3S_1$	1	3	1										-2
$1S_2$	1	0	0	1	0	0							1
$2S_2$	1	1	0	2	1	0							-2
$3S_2$	1	3	1	4	6	1							8
$1S_3$	1	0	0				1	0	0				1
$2S_3$	1	1	0				1	1	0				-1
$3S_3$	1	3	1				1	3	1				2
$1S_6$	1	0	0	1	0	0	3	0	0	1	0	0	-3
$2S_6$	1	1	0	2	1	0	9	9	0	6	1	0	18
$3S_6$	1	3	1	4	6	1	27	81	27	36	18	1	-216

We read off that $\mu_n(A_3, \Sigma_3) = \mu_n(1, C_2) = -2^{n-1}\mu_n(1, 1)$ (Lemma 3.14) and that $\mu_n(1, \Sigma_3) = -3^n\mu_n(1, 1)$. This last result shows that Lemma 3.18 does not in general extend to non-abelian groups.

4. EQUIVARIANT EULER CHARACTERISTICS OF G -POSETS

Let Π be a finite G -poset. The r th, $r \geq 1$, equivariant Euler characteristic of Π is [2] [7, Proposition 2.9]

$$\chi_r(\Pi, G) = \frac{1}{|G|} \sum_{X \in C_r(G)} \chi(\Pi^X) = \frac{1}{|G|} \sum_{A \leq G} \chi(\Pi^A) \varphi_r(A)$$

The first sum runs over the set $C_r(G)$ of all commuting r -tuples $X = (x_1, \dots, x_r)$ of elements of G . The second sum runs over all abelian subgroups A of G and $\varphi_r(A)$ is the number of generating r -tuples (a_1, \dots, a_r) of elements of A [5] [7, Remark 2.20].

We now specialize from general poset to posets of partitions. Let S be a finite G -set, $\Pi(S)$ the G -poset of partitions of G , and $\Pi^*(S) = \Pi(S) - \{\widehat{0}, \widehat{1}\}$ the G -poset of non-extreme partitions of S .

Definition 4.1. *The group G acts effectively on S if only the trivial element of G fixes all elements of S .*

Lemma 4.2. *Suppose that the abelian group A acts effectively on S . The following conditions are equivalent:*

- (1) *A acts isotypically on S*
- (2) *A acts freely on S*
- (3) *The degree of any non-identity element of A is $|S|$*
- (4) *The cycle structure of any element of A is d^m for some natural numbers d and m with $dm = |S|$*

If A acts isotropically on S then the order of A divides $|S|$.

Proof. If A acts isotypically and A is abelian, the isotropy subgroup at any point of S is the same subgroup, B , of A . The group B acts trivially on S , so B is the trivial subgroup since the action is effective. Thus A acts freely on S .

If A acts isotropically on S then $S = m1 \setminus A$ as right A -sets and $|S| = m|A|$. □

Lemma 4.3. *Let A be any abelian subgroup of Σ_n acting freely on $\Sigma_{n-1} \setminus \Sigma_n$. Put $m = \frac{n}{|A|}$.*

- (1) *The number of conjugates of A in Σ_n is*

$$|\Sigma_n : N_{\Sigma_n}(A)| = \frac{1}{|\text{Aut}(A)|} \frac{n!}{|A|^m m!}$$

- (2) *$\tilde{\chi}(\Pi^*(\Sigma_{n-1} \setminus \Sigma_n)^A) = (-1)^{m-1} \mu(1, A) |A|^{m-1} (m-1)!$ when $n \geq 2$.*
- (3) *$\tilde{\chi}(\Pi^*(\Sigma_{n-1} \setminus \Sigma_n)^A) |\Sigma_n : N_{\Sigma_n}(A)| = -(-1)^{n/|A|} \mu(1, A) \frac{1}{|\text{Aut}(A)|} (n-1)!$*

Proof. (1) It is a standard result that the normalizer of A in the right regular permutation representation of A is the holomorph $A \rtimes \text{Aut}(A)$ of A [8, pp 36–37]. Similarly, the normalizer of A in m times the right regular representation is $(A \wr \Sigma_m) \rtimes \text{Aut}(A)$ of order $|\text{Aut}(A)| |A|^m m!$.

(2) As an A -set $\Sigma_{n-1} \setminus \Sigma_n = m1 \setminus A$ consists of m free A -orbits. According to Lemma 3.18

$$\tilde{\chi}(\Pi^*(\Sigma_{n-1} \setminus \Sigma_n)^A) = \tilde{\chi}(\Pi^*(m1 \setminus A)^A) = \mu(1, A) |A|^{m-1} \mu_m(1, 1) = (-1)^{m-1} \mu(1, A) |A|^{m-1} (m-1)!$$

This formula also holds when A is trivial group. In this case, the left hand side is $\tilde{\chi}(\Pi^*(\Sigma_{n-1} \setminus \Sigma_n)) = (-1)^{n-1} (n-1)!$, and the right hand side is $(-1)^{n-1} (n-1)!$ as $\mu(1, 1) = 1$.

(3) This is an immediate consequence of (1) and (2). □

Proof of Theorem 1.1. on Combine the expression

$$\tilde{\chi}_r(\Pi^*(\Sigma_{n-1} \setminus \Sigma_n), \Sigma_n) = \frac{1}{n!} \sum_{\substack{[A \leq \Sigma_n] \\ A \text{ free and abelian}}} \tilde{\chi}(\Pi^*(\Sigma_{n-1} \setminus \Sigma_n)^A) \varphi_r(A) |\Sigma_n : N_{\Sigma_n}(A)|$$

for the r th equivariant Euler characteristic with Lemma 4.3.(3). Note also that any abelian group of order dividing n is realizable as a unique subgroup conjugacy class in the symmetric group Σ_n acting freely on $\Sigma_{n-1} \setminus \Sigma_n$. This gives

$$\tilde{\chi}_r(\Pi^*(\Sigma_{n-1} \setminus \Sigma_n), \Sigma_n) = -\frac{1}{n} \sum_{|A| \mid n} (-1)^{n/|A|} \mu(1, A) \frac{\varphi_r(A)}{|\text{Aut}(A)|}$$

where the sum ranges over the set of isomorphism classes of abelian groups A of order dividing n . The Möbius function $\mu(1, A)$ is completely known [5, 2.8]. Indeed, write $A = \prod A_p$ as the product of its Sylow p -subgroups A_p .

Then $\mu(1, A) = \prod \mu(1, A_p)$ and $\mu(1, A_p) = 0$ unless A_p is an elementary abelian p -group. For an elementary abelian p -group of rank d ,

$$\mu(1, C_p^d) = (-1)^d p^{\binom{d}{2}}$$

Suppose now that $A = \prod A_p$ where each Sylow p -subgroup $A_p = C_p^{d_p}$ is elementary abelian of rank d_p . By [6, Lemma 2.1], $\text{Aut}(A) = \prod_p \text{Aut}(A_p) = \prod_p \text{GL}_{d_p}(p)$ and clearly $\varphi_r(\prod A_p) = \prod \varphi_r(A_p)$. The number of surjections of C_p^r onto C_p^d is

$$\varphi_r(C_p^d) = \binom{r}{d}_p |\text{GL}_d(p)|$$

and consequently

$$\frac{\varphi_r(C_p^d)}{|\text{Aut}(C_p^d)|} = \binom{r}{d}_p$$

This finishes the proof. \square

Let $c_r(n) = (a * b_r)(n)$ denote Dirichlet convolution of the multiplicative arithmetic function $a(n)$ and $b_r(n)$. The function a is -1 ($+1$) on any even (odd) prime power and the multiplicative function b_r has value

$$(4.4) \quad b_r(p^e) = (-1)^e p^{\binom{e}{2}} \binom{r}{e}_p$$

on any prime power p^e .

Proposition 4.5. *The multiplicative arithmetic sequences b_r are given by $b_1 = \mu$ and the recurrence relations*

$$b_{r+1}(p^d) = p^d b_r(p^d) - p^{d-1} b_r(p^{d-1})$$

valid for all $r \geq 1$ and all prime powers p^d , $d \geq 0$.

Proof. Use Pascal's identities for ordinary and Gaussian binomial coefficients [9, Equation 17b]

$$\binom{d}{2} = \binom{d-1}{2} + (d-1), \quad \binom{r+1}{d}_p = p^d \binom{r}{d}_p + \binom{r}{d-1}_p$$

and the definition (4.4) of b_r . \square

In the following proposition, 1 is the constant sequence with value 1 on all $n \geq 1$.

Corollary 4.6. $(1 * b_{r+1})(n) = n b_r(n)$ for all $r, n \geq 1$.

Proof. The telescopic sum

$$(1 * b_{r+1})(p^d) = \sum_{e=0}^d b_{r+1}(p^e) = \sum_{e=0}^d (p^e b_r(p^e) - p^{e-1} b_r(p^{e-1}))$$

evaluates to $p^d b_r(p^d)$ at any prime power p^d . \square

Proposition 4.7. *The multiplicative arithmetic sequences c_r are given by $c_1 = 1, -2, 0, 0, \dots$ and*

$$c_{r+1}(n) = n(b_r(n) - b_r(n/2)) \quad (\text{where } b_r(n/2) = 0 \text{ for odd } n)$$

for all $r, n \geq 1$.

Proof. The two multiplicative sequences $c_1 = a * \mu$ and $1, -2, 0, 0, \dots$ are identical since they agree on all prime powers. For odd n , $c_{r+1}(n) = (a * b_{r+1})(n) = (1 * b_{r+1})(n) = n b_r(n)$ by Corollary 4.6. For powers of 2,

$$c_{r+1}(2^d) = (a * b_{r+1})(2^d) = b_{r+1}(2^d) - \sum_{e=0}^{d-1} b_{r+1}(2^e) = 2^d b_r(2^d) - 2^{d-1} b_r(2^{d-1}) - 2^{d-1} b_r(2^{d-1}) = 2^d (b_r(2^d) - b_r(2^{d-1}))$$

by the recurrence relation of Proposition 4.5. Thus $c_{r+1}(n) = n(b_r(n) - b_r(n/2))$ for even n by multiplicativity. \square

The multiplicative sequences c_r can be defined recursively. The initial sequence is $c_1 = 1, -2, 0, 0, \dots$. For $r \geq 1$,

$$c_{r+1}(2^d) = \begin{cases} 2c_r(2) & d = 1 \\ 2^d c_r(2^d) + \sum_{j=2}^d 2^{d+j-2} c_r(2^{d-j}) & d \geq 2 \end{cases}$$

for powers of 2. At powers of an odd prime p , $c_{r+1}(p^d) = p^d c_r(p^d) - p^{d-1} c_r(p^{d-1})$ as the sequences b_r and c_r coincide and we can refer to Proposition 4.5.

Corollary 4.8. *The Dirichlet series of the multiplicative arithmetic functions b_r and c_r are*

$$\sum_{n=1}^{\infty} \frac{b_r(n)}{n^s} = \frac{1}{\zeta(s)\zeta(s-1)\cdots\zeta(s-r+1)}, \quad \sum_{n=1}^{\infty} \frac{c_r(n)}{n^s} = \frac{2^s - 2}{2^s \zeta(s-1)\cdots\zeta(s-r+1)}$$

where $\zeta(s)$ is the Riemann ζ -function and $r \geq 1$.

Proof. Write $\beta_r(s)$ for the Dirichlet series of $b_r(n)$. Corollary 4.6 implies the recurrence

$$\zeta(s)\beta_{r+1}(s) = \beta_r(s-1)$$

as $nb_r(n)$, with series $\beta_r(s-1)$, is the Dirichlet convolution of 1, with series $\zeta(s)$, and $b_{r+1}(n)$. (The Dirichlet series of a Dirichlet convolution is the product of the Dirichlet series of the factors.) The expression for the Dirichlet series of $b_r(n)$ follows by induction starting with the series, $\zeta(s)^{-1}$, for $b_1 = \mu$. The Dirichlet series of the Dirichlet convolution $c_r = a * b_r$ is the product of this series and the series, $\zeta(s)(1 - 2^{1-s})$, of $a = 1 * c_1$. \square

It is easy to make explicit computations on a computer. The values of the multiplicative arithmetic function $\frac{1}{n}c_r(n) = \tilde{\chi}_r(\Pi^*(\Sigma_{n-1} \setminus \Sigma_n), \Sigma_n)$, $2 \leq n \leq 15$ and $1 \leq r \leq 5$, are

$\frac{1}{n}c_r(n)$	$n = 2$	3	4	5	6	7	8	9	10	11	12	13	14	15
$r = 1$	1	-1	0	0	0	0	0	0	0	0	0	0	0	0
$r = 2$	-2	-1	1	-1	2	-1	0	0	2	-1	-1	-1	2	1
$r = 3$	-4	-4	5	-6	16	-8	-2	3	24	-12	-20	-14	32	24
$r = 4$	-8	-13	21	-31	104	-57	-22	39	248	-133	-273	-183	456	403
$r = 5$	-16	-40	85	-156	640	-400	-190	390	2496	-1464	-3400	-2380	6400	6240

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